Cores of signed K_4-subdivisions

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Basic definitions

- A subdivision of a graph G is a graph G' which is obtained from G by *subdividing some or all edges of* G. The graph G has H-subdivision if G contains a subdivision of H as a subgraph.
- A minor of a graph G is a graph H obtained from G by a sequence of operators as *deleting vertices, deleting edges and contracting edges* in any order. The graph G has H-minor if it admits H as its minor.

Theorem

A graph G has K_4-minor if and only if G has K_4-subdivision. Generally, for H being a graph of degree at most 3, a graph G has H-minor if and only if G has H-subdivision.

Homomorphism of graphs

- Given two graphs G and H, a homomorphism of G to H is a mapping $\varphi : V(G) \to V(H)$ such that if $xy \in E(G)$, then $\varphi(x)\varphi(y) \in E(H)$.
- A core of graph G is the smallest subgraph of G to which G admits a homomorphism, denoted by $core(G)$. We say G is a core if it does not admit a homomorphism to any of its proper subgraphs.

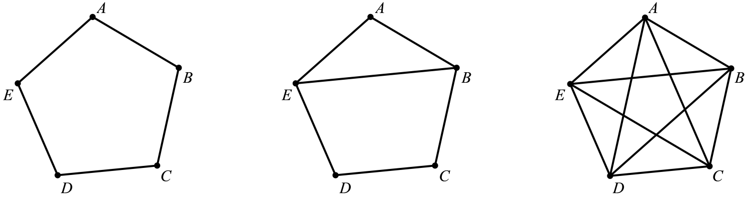


Figure: Examples

Signed graph

- A signed graph is a graph G together with an assignment of signs from the multiplicative group $\{+, -\}$ to its edges, denoted by (G, Σ), where the signature Σ stands for a set of edges assigned with $-$ for the graph G.
- Given a signed graph (G, Σ) and a vertex $v \in V(G)$, a resigning at v is an operator to *multiply the signs of all edges which are incident to v by $-$*. A signed graph (G, Σ) is a resigning of (G, Π) if it is obtained from (G, Π) by a sequence of resignings at vertices.
- We say (G, Σ) is equivalent to (G, Π) if (G, Σ) is a resigning of (G, Π).

Balance of cycle in the signed graph

The sign of a closed walk in signed graph is the product of signs of all edges in this closed walk. Especially, if a cycle is positive, we say it is balanced; if a cycle is negative, we say it is unbalanced.

Theorem (Zaslavsky,1982)

Two signed graphs (G, Σ_1) and (G, Σ_2) are equivalent if and only if they have the same set of unbalance cycles.

Homomorphism of signed graphs

Definition

A homomorphism of a signed graph (G, Σ) to (H, Π) is a mapping from the vertices and edges of G respectively to the vertices and edges of H such that adjacencies, incidences and signs of closed walks are preserved.

Definition

There exists a homomorphism of (G, Σ) to (H, Π) if there exists a signature Σ' such that $(G, \Sigma') \equiv (G, \Sigma)$ and there exists a two-edge-colored homomorphism of (G, Σ') to (H, Π).

Jaeger-Zhang conjecture

Conjecture (Jaeger, 1984)

If a graph G is $(4p+1)$-edge-connected, then there exists a $\mathbb{Z}_{2p+1}$-circular flow on G.

- This conjecture was proved for the case when the graph G is $(6p+1)$-edge-connected in 2013 by Lovász etc. Moreover, by the definition of odd-connectivity, they proposed a stronger result.
- In 2018, Han. etc have disproved this conjecture by giving a counterexample.

Jaeger-Zhang conjecture

Conjecture (Klostermeyer, W., & Zhang, C. Q., 2000)

If a planar graph G has odd-girth at least $4k+1$, then there exists a homomorphism of G to C_{2k+1}.

- For $k=1$, we have known it as Grötzsch's theorem.
- In 2000, Xuding ZHU showed that if the graph G has odd-girth $8k-3$, then $G \to C_{2k+1}$;
- In 2002, Borodin etc. proved the conjecture for odd-girth of G being $\frac{20k-2}{3}$;
- In 2013, from the view of flow, Lovász etc. gave the proof for this conjecture when the condition of odd-girth is $6k+1$.

Bipartite analogue of Jaeger-Zhang conjecture

Conjecture (Naserasr, R., Rollová, E., & Sopena, É. , 2013)

Let G be a bipartite and planar graph. If a signed graph (G, Σ) has unbalanced-girth at least $4k-2$, then there exists a homomorphism of (G, Σ) to an unbalanced cycle UC_{2k}.

K_4-subdivision

Suppose that G is a (a, b, c, a', b', c')-K_4, then exactly one of the following conditions holds:

- the core of G is K_2;
- the core of G is C_{2k+1} where $2k + 1$ is the odd girth of G;
- the core of G is itself, i.e., G is a core.

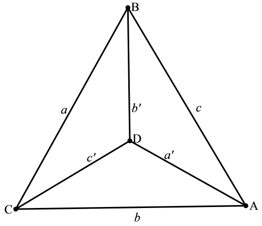


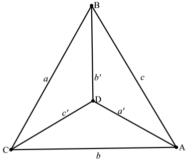
Figure: (a, b, c, a', b', c')-K_4

Conditions for K_4-subdivision being a core

Theorem

Let G be a (a, b, c, a', b', c')-K_4. Suppose that the odd girth of G is $a + b + c$. The graph G is a core if and only if the following conditions are all satisfied:

- *Every facial cycle is of odd length;*
- $b' + c' - a < a + b + c$;
- $a' + b' - c < a + b + c$;
- $a' + c' - b < a + b + c$.

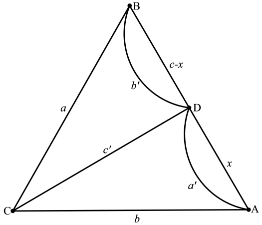


Let G be $(a, b, c, a', b', c') - K_4$. The graph G^x is obtained from G by identifying the center vertex D with one vertex in the outer facial cycle.

Lemma

Let G be $(a, b, c, a', b', c') - K_4$ with four odd facial cycles and $a + b + c$ be the odd girth of G. The graph G^x in the following figure admits a homomorphism to C_{a+b+c} if and only if this G^x satisfies one of following conditions:

- $a' \geq x$, $a' = x \pmod 2$, $b' \geq a + b + x$ *and* $c' \geq a + c - x$;
- $b' \geq c - x$, $a' \neq x \pmod 2$, $a' \geq a + b + c - x$ *and* $c' \geq b + x$.



A sketch of Proof

On one hand,

- If G admits a homomorphism to C_{a+b+c}, then there exists one value x such that after identifying D with one vertex in the outer facial cycle we obtain one G^x satisfying one of conditions in the previous lemma.
- Without loss of generality, such conditions will imply an inequality which is a contradiction to $b' + c' - a < a + b + c$.

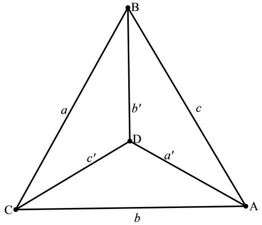


Figure: $(a, b, c, a', b', c') - K_4$

On the other hand, suppose that $b' + c' \geq 2a + b + c$, we have three cases:

- $a < b' \leq a + b$: We map D to one vertex in the edge AC and take $x = a + b - b'$.
- $a + b < b' \leq a + b + a'$: We map D to one vertex in the edge AB and take $y = b' - (a + b)$.
- $b' > a + b + a'$: There exists a homomorphism of b'-path to $(a + b + a')$-path. Then we could map the rest odd cycle to C_{a+b+c} directly.

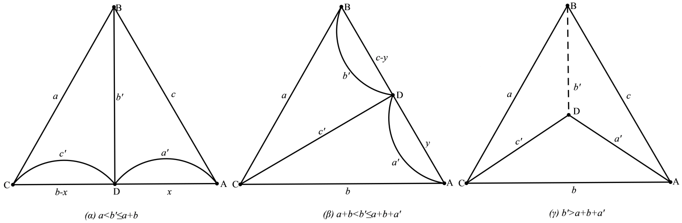
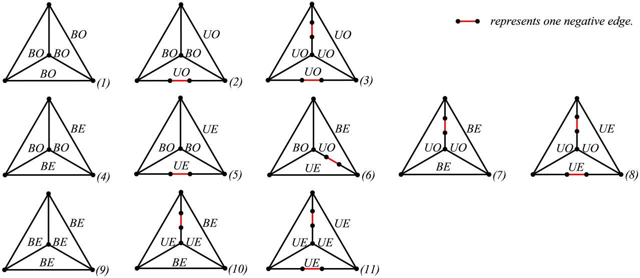


Figure: Cases

Classification of signed K_4-subdivision

- (4),(7),(9),(10) can not be cores;
- (1),(3) have same conditions for being cores; (11) has similar conditions as the previous two.
- (2),(5),(8) have similar conditions for being cores;(6) is one special case.



Signed bipartite K_4-subdivision

We define (G_a, Σ) to be the $(a+1, b, c, a'+1, b', c')$-$K_4$ as follows.

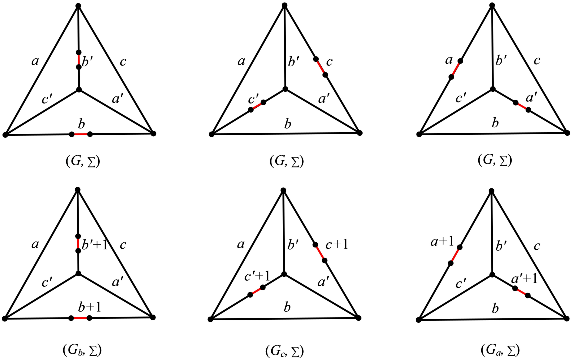


Figure: G_a, G_b, G_c

Lemma

Let (G,Σ) be a signed (a,b,c,a',b',c')-K_4 with four unbalanced even facial cycles(or four unbalanced odd facial cycles). The graph (G,Σ) admits a homomorphism to UC_{a+b+c} if and only if at least one of (G_a,Σ), (G_b,Σ), and (G_c,Σ) admits a homomorphism to $UC_{a+b+c+1}$.

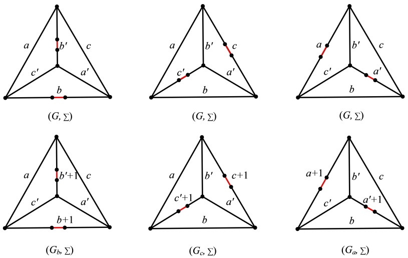


Figure: G_a, G_b, G_c

Signed bipartite K_4-subdivision

Theorem

Let G be a (a, b, c, a', b', c')-K_4 such that G is bipartite and Σ be a signature of G. Suppose that the unbalanced girth of G is $a + b + c$. Then the signed graph (G, Σ) is a core if and only if the following conditions are satisfied:

- *Every facial cycle is unbalanced;*
- $b' + c' - a < a + b + c$;
- $a' + b' - c < a + b + c$;
- $a' + c' - b < a + b + c$.

Homomorphism of signed K_4-minor-free graphs

Theorem

Let G be a K_4-minor free graph. All the signed graphs (G, Σ) admit homomorphisms to (H, Π) if and only if there exists a subgraph $(H', \Pi') \subset (H, \Pi)$, such that each edge of (H', Π') belongs to at least one positive triangle and one negative triangle.

Thank you for your attention. Any questions?

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